Advanced Calculus

Theorem Sheet

Axioms A1 (Basic Properties of $\mathbb{R}$):

1. Closure of addition and multiplication
2. Commutativity of addition
3. Associativity of addition
4. Existence of an additive identity
5. Existence of an additive inverse
6. Commutativity of multiplication
7. Associativity of multiplication
8. Existence of a multiplicative identity
9. Existence of multiplicative inverses
10. The Distributive Property
11. The Nontriviality Assumption

Theorems T2 (Basic Properties of $\mathbb{R}$):

1. The additive identity, 0 is unique.
2. $a0 = 0a = 0$
3. $ab = 0 \Rightarrow a = 0$ or $b = 0$
4. The equation $a + x = 0$ has a solution.
5. The solution to the above equation is unique.
6. The multiplicative identity is unique.
7. $a \neq 0 \Rightarrow ax = 1$ has a solution.
8. The solution to the above equation is unique.
9. $-(-a) = a$
10. $a \neq 0 \Rightarrow (a^{-1})^{-1} = a$
11. $a \neq 0 \Rightarrow (-a^{-1}) = -a^{-1}$

Axioms A3 (Positivity Axioms):

1. a, b are positive $\Rightarrow ab$ and $a + b$ are positive.
2. Exactly one of the following is true
   - a is positive
   - $-a$ is positive
   - $a = 0$
3. $a > b$ means $a - b$ is positive.
4. $a > 0$ means a is positive
5. $a \geq b$ means $a - b$ is positive or zero.

Theorems T4 (Positivity Properties):

1. $a \neq 0 \Rightarrow a^2 > 0$
2. $1 > 0$
3. $a > 0 \Rightarrow a^{-1} > 0$
4. $c > 0$ and $a > b \Rightarrow ac > bc$
5. $c < 0$ and $a > b \Rightarrow ac < bc$

Theorems T5 (Induction Theorems):

1. Theorem: $\mathbb{N}$ is inductive
2. If $A \subseteq \mathbb{N}$ is inductive, then $A = \mathbb{N}$.
3. Let $S(n)$ be a statement (claim) based on the natural number n. Assume the following are true:
   - $S(1)$
   - $S(k) \Rightarrow S(k+1)$

   Then $S(n)$ is true for every natural number n.

Theorems T6 (Theorems on numbers):

1. $n, m \in \mathbb{N} \Rightarrow n + m \in \mathbb{N}$
2. $n, m \in \mathbb{N} \Rightarrow nm \in \mathbb{N}$
3. If $x \in \mathbb{Q}$, then there are some $m, n \in \mathbb{Z}$ with at least one of them odd such that $x = \frac{m}{n}$
4. If $n \in \mathbb{Z}$ is even, then n^2 is as well.

Axioms A7 (Sup exists): Every set of real numbers that has an upper bound, has a single smallest upper bound.

Theorem T8 ($\sqrt{x}$ exists): Let c be a positive number. There is a unique solution to the system below.

$$x > 0$$
$$x^2 = c$$

Theorems T9 (Archimedean Property):

1. $\forall_{c>0} \exists_{n \in \mathbb{N}} (n > c)$
2. $\forall_{\varepsilon>0} \exists_{n \in \mathbb{N}} \left(\frac{1}{n} < \varepsilon\right)$

Theorem T10: Let $n \in \mathbb{Z}$. There is no integer in the interval $(n, n+1)$

Theorem T11: Assume $\emptyset \neq S \subseteq \mathbb{Z}$, and that S is bounded above. Then S has a maximum element.

Theorem T12: $\forall_{c \in \mathbb{R}} \exists!_{k \in \mathbb{Z}} \left(k \in [c, c+1)\right)$

Theorem T13: $\mathbb{Q}$ is dense in $\mathbb{R}$.

Theorems T14: For $x \in \mathbb{R}$, $d > 0$:

1. $|x| \leq d$ iff $-d \leq x \leq d$
2. $-|x| \leq x \leq |x|$

Theorem T15 (The Triangle Inequality): For all real a, b: $|a + b| \leq |a| + |b|$

Theorem T16 (The Reverse Triangle Inequality): For all real a, b: $\big||a| - |b|\big| < |a - b|$

Theorem T17: Fix $a \in \mathbb{R}$ and $r > 0$. TFAE:

- $|x-a| < r$
- $a-r < x < a+r$
- $x \in (a-r, a+r)$

Theorem T18: Let $a, b \in \mathbb{R}$, $n \in \mathbb{N}$. Then:

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$$

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^{(n-1)-k} b^k$$

Theorem T19 (Finite geometric series): Let $m \in \mathbb{N}; r \neq 1$. Then:

$$1 + r + r^2 + \cdots + r^m = \frac{1-r^{m+1}}{1-r}$$

$$\sum_{k=0}^{m} r^k = \frac{1-r^{m+1}}{1-r}$$

Theorem T20 (Binomial Theorem): $a, b \in \mathbb{R},\ n \in \mathbb{N}$. Then:

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \cdots + \binom{n}{n} b^n$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Lemma L21: If $\{a_n\} \to 0$ and $\exists_{N\in\mathbb{N}} \forall_{n\geq N} (|b_n| \leq C|a_n|)$ then also $\{b_n\} \to 0$.

Lemma L22: If $\{a_n\} \to a$ and $\exists_{N\in\mathbb{N}} \forall_{n\geq N} (|b_n - b| \leq C|a_n - a|)$ then also $\{b_n\} \to b$.

Theorem T23 (Sum property for convergence): Assume $\{a_n\} \to a$ and $\{b_n\} \to b$. Then $\{a_n + b_n\} \to a + b$.

Lemma L24: Assume $\{a_n\} \to a$, then $\{ca_n\} \to ca$.

Lemma L25: Assume $\{a_n\} \to 0$ and $\{b_n\} \to 0$, then also $\{a_n b_n\} \to 0$.

Theorem T26 (product property for convergence): Assume $\{a_n\} \to a$ and $\{b_n\} \to b$. Then $\{a_n b_n\} \to ab$.

Theorem T27: Assume $b_n \neq 0,\ b \neq 0$, and $\{b_n\} \to b$. Then $\left\{\frac{1}{b_n}\right\} \to \frac{1}{b}$.

Theorem T28 (Quotient property for convergence): Assume $b_n \neq 0$, $b \neq 0$, $\{a_n\} \to a$, and $\{b_n\} \to b$. Then $\left\{\frac{a_n}{b_n}\right\} \to \frac{a}{b}$

Theorem T29 (Linearity property of convergence): Assume $\{a_n\} \to a$, and $\{b_n\} \to b$. Then $\{ca_n + db_n\} \to ca + db$

Theorem T30 (Polynomial property for convergence): Assume $\{a_n\} \to a$, and $f(x)$ is a polynomial. Then the polynomial of the sequence also converges: $\{f(a_n)\} \to f(a)$

Theorem T31 (Monotone Convergence Theorem): Let $\{a_n\}$ be a monotone sequence. Then $\{a_n\}$ converges if and only if it is bounded. Furthermore, if it does converge, it converges to either its sup or inf.

Theorem T32 (Nested Interval Theorem): Construct a sequence of intervals $I_n \coloneqq [a_n, b_n]$ that are nested, by which we mean $\forall_{n\in\mathbb{N}}(I_{n+1} \subseteq I_n)$.
If $\{b_n - a_n\} \to 0$, then for some $c \in \mathbb{R}$:

$$\{a_n\} \to c$$
$$\{b_n\} \to c$$
$$\bigcap_{n=1}^{\infty} I_n = \{c\}$$

Theorem T33: Let $\{a_n\}$ be a sequence and assume $\{a_n\} \to a$. Then every subsequence also converges to a: $\{a_{n_k}\} \to a$

Theorem T34: Every sequence has a monotone subsequence.

Theorem T35: Every bounded sequence has a convergent subsequence.

Theorem T36: **(Sequential Compactness of closed intervals):** $[a, b]$ is sequentially compact for all $a < b$.

Theorem T37: Let $S \subseteq \mathbb{R}$. The following are equivalent:
1. S is closed and bounded.
2. S is sequentially compact
3. S is compact

Theorem T38: Let $f, g: D \to \mathbb{R}$ both be continuous functions. Then $f + g, f - g$, and $f \cdot g$ are also continuous.

Theorem T39: Let $f, g: D \to \mathbb{R}$ both be continuous functions. Assume $g(x) \neq 0$ on D. Then $\frac{f}{g}$ is continuous.

Corollary C40: Let $p, q: \mathbb{R} \to \mathbb{R}$ be polynomials. Then p and q are continuous, as well as the rational function $\frac{p}{q}: D \to \mathbb{R}$ where $D = \{x \in \mathbb{R} | q(x) \neq 0\}$.

Theorem T41: Let $f: D \to \mathbb{R}$ and $g: U \to \mathbb{R}$. Assume the following.
- $f(D) \subseteq U$
- f is continuous at $x_0 \in D$
- g is continuous at $f(x_0) \in U$.

Then $g \circ f$ is continuous at x_0.

Theorem T42 (Extreme Value Theorem): Let $f:[a,b] \to \mathbb{R}$ be continuous. Then f attains both a maximum and minimum value.

Theorem T43 (Intermediate Value Theorem): Let $f:[a,b] \to \mathbb{R}$ be continuous. Let $c \in \mathbb{R}$ such that $f(a) < c < f(b)$. Then there is some $x_0 \in (a,b)$ such that $f(x_0) = c$. The same is true if we replace each "<" with ">".

Theorem T44: Let I be an interval and $f: I \to \mathbb{R}$ be continuous. Then $f(I)$ is also an interval.

Theorem T45: Let $f:[a,b] \to \mathbb{R}$ be continuous. Then f is also uniformly continuous.

Theorem T46: Let $f: D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$. The sequential definition of continuity at $x_0 \in D$ is equivalent to the $\varepsilon - \delta$ criterion of continuity at x_0.

Theorem T47: Let $f, g: D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$ both be differentiable. Then $f + g$ and fg are also differentiable Also, $\frac{f}{g}$ is differentiable on $\{x \in D | g(x) \neq 0\}$.

Theorem T48: Let $f: D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$ be differentiable. Then f is continuous on D.

Theorem T49: Let P be a partition of $[a,b]$ and P_2 be a refinement of P. Then $L(f,P) \leq L(f,P_2)$ and $U(f,P_2) \leq U(f,P)$

Theorem T50: Let P_1 and P_2 be partitions of $[a,b]$. Then $L(f,P_1) \leq U(f,P_2)$

Theorem T51: Let $a < b$ and $f:[a,b] \to \mathbb{R}$ be a function. Then $\underline{\int_a^b f} \leq \overline{\int_a^b f}$.